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Small-Angle Multiple Scattering of Fast Charged Particles Using Molière Screening*

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The Dalitz formula for the second Born approximation for the elastic-scattering cross section of an electron by a Yukawa potential is extended to a potential represented by a sum of three such terms. This result is used to obtain the distribution function for multiple scattering following the method of Bethe and Nigam, Sundaresan, and Wu. The $1/e$ widths of the distribution function thus obtained are compared to the experimental results of Hanson, Lanzl, Lyman, and Scott.

I. INTRODUCTION

THE theory of small-angle multiple scattering of fast charged particles is of interest in discussions of such phenomena as the interactions of high-energy electrons and mesons with a finite thickness of matter. Goudsmit and Saunderson[1] have given an "exact" theory of multiple scattering based on the addition theorem for spherical harmonics and a knowledge of the distribution of scattering for a single collision. Molière's[2] theory of multiple scattering of electrons and other charged particles has been shown by Bethe[3] to have a close quantitative relation to that of Goudsmit and Saunderson. Nigam, Sundaresan, and Wu[4] have modified Molière's theory by the use of the improved single-scattering cross section of Dalitz.[5] Dalitz has calculated the second Born approximation for scattering of a Dirac electron by a Yukawa potential. Consequently, Nigam *et al.* were constrained to use a potential of the form

$$V(r)=(Ze/r)e^{-\lambda r}. \tag{1}$$

Molière, however, used a Thomas-Fermi potential

$$V(r)=(Ze/r)\omega(r/a) \tag{2}$$

with the Thomas-Fermi function represented as

$$\omega(r/a)=\sum_{i=1}^{3} a_i e^{-b_i r/a} \tag{3}$$

and $a=0.466\times10^{-8}\ Z^{-1/3}$ cm. Molière employs the coefficients

$$a_1=0.1 \qquad a_2=0.55 \qquad a_3=0.35$$
$$b_1=6 \qquad b_2=1.2 \qquad b_3=0.3. \tag{3a}$$

He states that the error between this representation and the actual Thomas-Fermi function is less than 0.002 for $0\leq r/a\leq 6$. Nigam *et al.* assert that the Thomas-Fermi potential (2) can be well represented by (1) if one introduces a parameter μ by the relation

$$\lambda=\mu\lambda_0, \tag{4}$$

where $\lambda_0=Z^{1/3}/0.885a_0$ is the reciprocal of the Thomas-Fermi length, a, and a_0 is the radius of the first Bohr orbit. This assertion has been questioned by Scott,[6] and is the basis of this communication.

II. SECOND BORN APPROXIMATION FOR SINGLE SCATTERING

Mitra[7] has extended the work of Vachaspati[8] and Lewis[9] to obtain an expression for the single-scattering cross section, up to second order in the Born approximation, for the Rozental[10] representation of the Thomas-Fermi potential. This representation is of the same form as (3) except that the coefficients are given by

$$a_1=0.164 \qquad a_2=0.581 \qquad a_3=0.255$$
$$b_1=4.356 \qquad b_2=0.947 \qquad b_3=0.246. \tag{3b}$$

We have independently followed the work of Dalitz to obtain the differential elastic-scattering cross section resulting from a potential of the form

$$V(r)=\frac{Ze}{r}\sum_{i=1}^{3} a_i e^{-\lambda_i r}, \tag{5}$$

where

$$\lambda_i=b_i\lambda \tag{6}$$

with the a_i and b_i as given by (3a) or (3b) and λ has the same meaning as in (4). Thus the potential (5) reduces to Molière's representation of the Thomas-Fermi potential (2) if $\mu=1$. The general expression for the differential cross section corresponding to (2.5) and

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(2.6) of Dalitz has the form

$$\frac{d\sigma}{d\Omega}=4Z^2e^4E^2\left[\sum_{i=1}^{3}\frac{a_i}{\lambda_i^2+4p^2\sin^2(\theta/2)}\right]^2$$

$$\times\left\{\left(1-v^2\sin^2\frac{\theta}{2}\right)\left[1-\frac{Ze^2E}{\pi^2}\frac{\sum_{j=1}^{3}\sum_{k=1}^{3}a_ja_k\,\mathrm{Re}(I^{jk}+J^{jk})}{\sum_{i=1}^{3}\frac{a_i}{\lambda_i^2+4p^2\sin^2(\theta/2)}}\right]-\frac{m^2}{E}\frac{Ze^2}{\pi^2}\frac{\sum_{j=1}^{3}\sum_{k=1}^{3}a_ja_k\,\mathrm{Re}(I^{jk}-J^{jk})}{\sum_{i=1}^{3}\frac{a_i}{\lambda_i^2+4p^2\sin^2(\theta/2)}}\right\} \quad (7)$$

where

$$\mathrm{Re}(I^{jk}\pm J^{jk})=\frac{-\pi^2\left[1\pm\frac{(\lambda_j^2+\lambda_k^2+4p^2)}{4p^2\cos^2(\theta/2)}\right]\tan^{-1}\frac{N_{jk}}{D_{jk}}}{p\{(\lambda_j^2-\lambda_k^2)^2+4\sin^2(\theta/2)[2p^2(\lambda_j^2+\lambda_k^2)+\lambda_j^2\lambda_k^2]+16p^4\sin^4(\theta/2)\}^{1/2}}$$

$$\mp\frac{\pi^2}{4p^3\cos^2(\theta/2)}\left\{\tan^{-1}\frac{2p(\lambda_j+\lambda_k)}{4p^2-\lambda_j\lambda_k}-\frac{1}{\sin(\theta/2)}\tan^{-1}\frac{\eta_{jk}}{\delta_{jk}}\right\} \quad (8)$$

and

$$N_{jk}=2p(\lambda_j+\lambda_k)\left[(\lambda_j-\lambda_k)^2+4p^2\sin^2\frac{\theta}{2}\right]\left\{(\lambda_j^2-\lambda_k^2)+4\sin^2\frac{\theta}{2}[2p^2(\lambda_j^2+\lambda_k^2)+\lambda_j^2\lambda_k^2]+16p^4\sin^4\frac{\theta}{2}\right\}^{1/2}$$

$$D_{jk}=(4p^2+\lambda_j\lambda_k)(\lambda_j^2-\lambda_k^2)^2+16p^4\sin^4\frac{\theta}{2}(4p^2-\lambda_j\lambda_k)+16p^2\sin^2\frac{\theta}{2}[2p^2(\lambda_j^2+\lambda_k^2)+\lambda_j^2\lambda_k^2]$$

$$\eta_{jk}=4p\sin\frac{\theta}{2}(\lambda_j+\lambda_k)\left[4p^2\sin^2\frac{\theta}{2}+(\lambda_j-\lambda_k)^2\right]$$

$$\delta_{jk}=16p^2\sin^2\frac{\theta}{2}\left(p^2\sin^2\frac{\theta}{2}-\lambda_j\lambda_k\right)-(\lambda_j^2-\lambda_k^2)^2.$$

Mitra has given numerical results only for 90° scattering and for four values of Z using the coefficients of (3b). Although his results were not presented in a form readily amenable to multiple-scattering calculations, his general formulas, apart from misprints, appear to agree with our results (7) and (8).

III. DISTRIBUTION FUNCTION FOR MULTIPLE SCATTERING

Following Goudsmit and Saunderson and Nigam *et al.*, we write the angular distribution function as

$$f(\theta,t)=\sum_{l=0}^{\infty}(l+\tfrac{1}{2})P_l(\cos\theta)\exp\{-[Q_l'+Q_l'']\}, \quad (9)$$

where

$$Q_l'+Q_l''=2\pi Nt\int_0^{\pi}\left(\frac{d\sigma}{d\Omega}\right)[1-P_l(\cos\varphi)]\sin\varphi d\varphi. \quad (10)$$

As usual, the actual total path of the scattered particle is taken to be the same as the target foil thickness, t, and N is the density of scattering atoms. After substitution of (7) into (10) one obtains

$$Q_l'+Q_l''=\tfrac{1}{2}\chi_c^2\left\{l(l+1)\left[\ln\frac{2}{\chi_\alpha}-\frac{1}{2}+(1-\psi_{(l)}-C)\cdot\left(\sum\sum a_ia_m+\frac{\alpha}{\beta}(1-\beta^2)\sum\sum\sum a_ia_ja_k(\chi_j+\chi_k)\right)\right]-(\psi_{(l)}+C)[\beta^2\sum\sum a_ia_m+\pi\alpha\beta\sum\sum\sum a_ia_ja_k]+2\pi\alpha\beta l\sum\sum\sum a_ia_ja_k\right\}, \quad (11)$$

where

$$\chi_i=\hbar\lambda_i/p,$$
$$\chi_c^2=4\pi Nte^4z^2Z(Z+1)/p^2v^2,$$
$$\alpha=zZe^2/\hbar c,$$
$$\beta=v/c,$$
$$\psi_{(l)}+C=1+\tfrac{1}{2}+\cdots+(1/l),$$
$$C=0.57721\cdots,$$

TABLE I. Multiple scattering of electrons by Au and Be foils. [Hanson *et al.* (1951)].

Material	Be	Au
Experiment:		
ρt (mg/cm^2)	491.3	37.28
E (MeV)	15.24	15.67
$\theta_{1/e}$ (deg)	4.25	3.76
$\mu=1.00$ $\chi_c(B)^{1/2}$	4.87	4.40
$\theta_{1/e}$	4.61	4.14
$\theta_{1/e}$ (Molière)	4.57	3.83
$\mu=1.12$ $\chi_c(B)^{1/2}$	4.81	4.33
$\theta_{1/e}$	4.55	4.06
$\theta_{1/e}$ (Nigam *et al.*)	4.60	4.10
$\mu=1.80$ $\chi_c(B)^{1/2}$	4.55	4.01
$\theta_{1/e}$	4.28	3.72
$\theta_{1/e}$ (Nigam *et al.*)	4.35	3.80

and

$$\ln\frac{2}{\chi_\alpha}-\tfrac{1}{2}=\sum\sum a_i a_m A_{im}+\ln\frac{2}{k}\sum\sum a_i a_m$$
$$+\frac{\alpha}{\beta}\ln\frac{2}{k}\sum\sum\sum a_i a_j a_k(\chi_j+\chi_k)$$
$$+\frac{\alpha}{\beta}\sum\sum\sum a_i a_j a_k(\chi_j+\chi_k)D_{ijk}$$
$$-\alpha\beta(1+\ln 2)\sum\sum\sum a_i a_j a_k(\chi_j+\chi_k)$$
$$-\frac{\alpha\beta\pi^2}{4}\sum\sum\sum a_i a_j a_k\chi_i+\alpha\beta\sum\sum\sum a_i a_j a_k\chi_i B_{ijk}$$
$$+\alpha\beta\sum\sum\sum a_i a_j a_k(\chi_j+\chi_k)\ln(\chi_j+\chi_k).\quad(12)$$

The quantity k appearing in (10) has the same meaning as in Nigam *et al.* and

$$A_{im}=\ln\frac{k}{\chi_0}-\frac{b_m^2\ln b_m-b_i^2\ln b_i}{b_m^2-b_i^2},$$
$$B_{ijk}=\int_0^\infty\frac{\tan^{-1}\left[\left(\frac{b_j+b_k}{b_i}\right)t\right]dt}{t^2+1},$$
$$D_{ijk}=\ln\frac{k}{\chi_0}-\frac{(b_j+b_k)^2\ln(b_j+b_k)-b_i^2\ln b_i}{(b_j+b_k)^2-b_i^2}.$$

The b_i are the same as in (6) and

$$\chi_0=\hbar\lambda/p.\quad(13)$$

Using the set of a_i and b_i given by Molière, the sums in (11) and (12) can be performed and B_{ijk} numerically integrated to yield

$$Q_l'+Q_l''=\frac{1}{2}\chi_c^2\left\{l(l+1)\left[\ln\frac{2}{\chi_\alpha}-\frac{1}{2}+(1-\psi_{(l)}-C)\cdot\left(1+2.73\chi_0\frac{\alpha}{\beta}(1-\beta^2)\right)\right]-(\beta^2+\pi\alpha\beta)\cdot[\psi_{(l)}+C]+2\pi\alpha\beta l\right\}\quad(14)$$

TABLE II. Some numerical values for the distribution function for gold.

$\theta/\chi_c(B)^{1/2}$	$f^{(0)}+\frac{1}{B}(f^{(1)\prime}+f^{(1)})$: $\mu=1.00$ ($B=8.0693$)	$\mu=1.12$ ($B=7.8090$)	$\mu=1.80$ ($B=6.7029$)
0.00	2.0437	2.0462	2.0591
0.90	0.8143	0.8120	0.8005
0.95	0.7352	0.7329	0.7210
1.00	0.6607	0.6584	0.6464

which is of the same form as Nigam *et al.* (56a). The difference in potentials between the present work and that of Nigam *et al.* is manifested through the term

$$\ln\frac{1}{\chi_\alpha}=\ln\frac{1}{\chi_0}-0.079-2.73\chi_0\frac{\alpha}{\beta}(\ln\chi_0+0.772)+2.73\chi_0\alpha\beta(\ln\chi_0-1.043).\quad(15)$$

IV. EXPERIMENTAL RESULTS

The experimental results of Hanson, Lanzl, Lyman, and Scott[11] have been discussed by Scott[6] and by Nigam *et al.* For these experiments the unnormalized distribution function can be written as

$$f(\theta,t)=f^{(0)}+\frac{1}{B}(f^{(1)\prime}+f^{(1)})+\frac{1}{2!B^2}(f^{(2)\prime}+f^{(2)})+\cdots,\quad(16)$$

where B is a parameter having the same meaning as in Nigam *et al.* and

$$f^{(0)}=\int_0^\infty J_0(\varphi u)e^{-u^2/4}u\,du\quad(17)$$

$$f^{(1)}=\int_0^\infty J_0(\varphi u)e^{-u^2/4}\frac{u^2}{4}\ln\frac{u^2}{4}u\,du\quad(18)$$

$$f^{(1)\prime}=-\pi\alpha\beta\chi_c(B)^{1/2}\int_0^\infty J_0(\varphi u)e^{-u^2/4}u^2\,du\quad(19)$$

$$\varphi=\theta/\chi_c(B)^{1/2}.\quad(20)$$

The integrals in (17) and (18) have been evaluated by Watson[12] and Molière, respectively, and $f^{(1)\prime}$ can be evaluated in terms of the confluent hypergeometric function $F(\frac{3}{2};1;-\varphi^2)$. Scott gives a rather extensive table of these and associated functions from which our Table II can be extended considerably. Nigam *et al.* quote a value of $\mu=1.12$ based on the Thomas-Fermi field. However, this value of μ yields a calculated $1/e$ width which does not agree very well with the experi-

[11] A. O. Hanson, L. H. Lanzl, E. M. Lyman, and M. B. Scott, Phys. Rev. **84**, 634 (1951).

[12] G. N. Watson, *A Treatise on the Theory of Bessel Function*, (Cambridge University Press, Cambridge, England, 1948), p. 69.

mental results. They then empirically chose a value of $\mu=1.80$ based on the experimental results of Au and found that this same value of μ gives good results for Be. For the potential of (5) to reduce to Molière's representation of the Thomas-Fermi potential $\mu=1$. We have calculated the $1/e$ width corresponding to the experiments of Hanson *et al.* for all three values of μ. The results are shown in Table I where we have compared the $\theta_{1/e}$ of Molière (for $\mu=1.00$) and Nigam *et al.* (for $\mu=1.12$ and $\mu=1.80$) as listed in Table II of Nigam *et al.* Scott[6] has recalculated the results of Nigam *et al.* and obtains values very close to ours. He obtains $\theta_{1/e}=4.05°$ and $3.71°$ for Au and $4.50°$ and $4.21°$ for Be with $\mu=1.12$ and 1.80, respectively, compared to our results of 4.06° and 3.72° for Au and 4.55° and 4.28° for Be. In Table II some numerical values of the first two terms of (16) are given. The fact that the Thomas-Fermi function falls off so slowly with distance and hence is unrealistic for small-angle scattering probably accounts for the lack of agreement with experiment for $\mu=1$. The agreement of our results with those of the Nigam *et al.* theory (as recalculated by Scott) is due to the similarity in functional form between (1) and (5).

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Functional Analysis and Scattering Theory*

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We formulate the nonrelativistic scattering problem as an integral equation with a kernel which is completely continuous for all energies. We then are able to give a rigorous justification for the Fredholm method, quasiparticle method, and, for weak enough interactions, the Born expansion. We also give an explicit lower bound for the radius of convergence of the Born series and of the Born series modified by the introduction of quasiparticles. We furthermore show that all these expansions coverage uniformly in the physical region of energy and momentum transfer.

I. INTRODUCTION

THIS paper is concerned with the application of functional analysis to the problem of scattering of a single nonrelativistic particle by a fixed interaction V. Our purpose when we began this work was to provide a rigorous justification for the "quasiparticle method" presented by one of us in previous papers.[1,2] The sticky point was that the scattering kernel $[W-H_0]^{-1}V$ is not even bounded in the physical scattering region $W\geqq 0$, though it is L^2 for all other W. We overcome this problem here by using a new "symmetrized" kernel[3]

$$V^{\frac{1}{2}}[W-H_0]^{-1}V^{\frac{1}{2}},$$

which is L^2 for all W. (Sec. II and IV.)

Having solved our original problem in this way, we were pleased to find a number of useful by-products:

(1) We give an explicit lower bound on the radius of convergence of the ordinary Born series for all energies. This had previously been done for the bound-state problem[2] but not for the scattering problem (Sec. III). In fact, we give explicit upper bounds on the nth order terms of the Fredholm and Born series (Sec. V), which should be useful for practical calculations.

(2) We do the same for the Born series modified by the introduction of a "quasiparticle," so that it is possible to be certain that the modified Born series converges (Sec. IV).

(3) We show that all these expansions [Fredholm, quasi-Born, and, for weak enough interactions, ordinary Born] converge *uniformly* in the physical region of energy and momentum transfer (Sec. V).

Most of our work is applicable to very general interactions, but we give special attention to the case of a local (not necessarily central) potential $V(\mathbf{r})$, subject

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[1] S. Weinberg, Phys. Rev. **130**, 776 (1963).

[2] S. Weinberg, Phys. Rev. **131**, 440 (1963).

[3] While this paper was being written we received a preprint by F. Coester, Phys. Rev. **133**, B1516 (1964), who uses essentially the same idea. He factors the potential as $V=V^{(+)}V^{(-)}$, and studies the kernel $V^{(-)}[W-H_0]^{-1}V^{(+)}$. The advantage in our choosing $V^{(+)}=V^{(-)}=V^{\frac{1}{2}}$ is that it minimizes the L^2 norm of the scattering kernel, thereby giving a superior lower bound on the radius of convergence of the Born series. The first author to use $V^{\frac{1}{2}}$ symmetrization appears to be J. Schwinger [Proc. Natl. Acad. Sci. U. S. **47**, 122 (1961)], who employed it to study the bound state problem. The general idea of performing similarity transformations on the scattering kernel has also been discussed by L. Brown, D. Fivel, B. W. Lee, and R. Sawyer [Ann. Phys. (N. Y.) **23**, 167 (1963)]. These last authors concentrate on the kernel $[W-H_0]^{-\frac{1}{2}}V[W-H_0]^{-\frac{1}{2}}$; in this connection see also footnote 8 of Ref. 2 and B. Lee, in *Theoretical Physics* [International Atomic Energy Agency, Vienna, 1963], p. 331. However, the kernel $[W-H_0]^{-\frac{1}{2}}V[W-H_0]^{-\frac{1}{2}}$ is not L^2 for $W>0$.